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THE MAGNETIC ELASTIC DISPERSION

Jorge Páez
Escuela de Física.
Universidad de Costa Rica.
San José. Costa Rica.

Liudmila Semionova
Departamento de Física.
Universidad Nacional.
Heredia. Costa Rica.

RESUMEN

Se ha efectuado un estudio del "scattering" elástico de un electrón en un campo magnético homogéneo constante. Se evidencia cómo la sección transversal depende de la orientación del "spin" del electrón. Si la polarización del "spin" se efectúa a lo largo o contra el campo externo, entonces la sección transversal es cuatro veces mayor que ésa, cuando la polarización del "spin" apunta en la dirección de su propio movimiento.

ABSTRACT

We present a study of the elastic scattering of an electron in a constant homogeneous magnetic field. The dependency of the cross section on the electron spin orientation is given. If the spin polarization is along or against the external field, then the cross section is four times larger than the cross section if the spin polarization pointed along its own motion.

1. INTRODUCTION

The quantum mechanical study of the motion of fermions in a magnetic field has some theoretical and experimental interesting aspects that have called the attention of physicists over and over again [1]. Since the early days of the relativistic quantum mechanics we know that there is an analytical solution of the Dirac equation for an electron in an homogeneous constant magnetic field both in cartesian and in cylindrical coordinates [2]

The motion of a charged particle in a constant magnetic field is treated in both relativistic and non-relativistic quantum theory by Johnson and Lippmann[3]. By knowing the wave function of the electron in an homogeneous magnetic field one is able to calculate the transition rates of processes such as magnetic Bremsstrahlung[4], pair-creation and pair-anihilation[5] in magnetic fields, quantum modifications to the magnetic Bremsstrahlung[6], gravitational and electromagnetic radiation of relativistic particles[7], polarization and spin effects in synchrotron radiation[8], transitions to the ground state in synchrotron radiation[9], etc.

The existence of magnetic fields of the order of $10^{12} - 10^{13}$ Gauss in the vicinity of pulsars[10] has developed interest among astrophysicists in the effects of superstrong magnetic fields $B \sim 10^{12}$ Gauss in the elementary processes of quantum electrodynamics.

The processes of synchrotron and cyclotron emission may be responsible for the hard radiation observed from a variety of collapsed stellar objects[11]. An understanding of the physics under such conditions implies a knowledge of the various quantum electrodynamical processes which occur in such an intense field. As we have pointed out above, much work has been done in this direction. However, the case to be discussed in this paper is the elastic magnetic scattering[12]. We work in the Furry framework[13] where the S-matrix is

$$S_{fi} = T \exp\left(-ie \int : \bar{\Psi}_{in}(x)\, \gamma_\mu \cdot \Psi_{out}(x)\, A_\mu^{ext}(x)\, d^4 x :\right),$$

where $\Psi(x)$ is the wave function of the electron in presence of A_μ^{ext} and it is the solution of the Dirac equation[14]

$$(\gamma^\mu (i\partial_\mu - \epsilon A_\mu^{ext}) - x)\, \Psi_e(x) = 0$$

If we expand the S–matrix as in the usual perturbation theory (which is an expansion in α), we can describe the matrix elements in terms of Feynman's diagrams. However, we introduce double lines for the fermions in order to differenciate this perturbation theory from the usual Q.E.D.[15]. We want to emphasize in this way that we are using the solutions of the Dirac equation and not plane waves. For the case to be treated here we neglect the anomalous magnetic moment solution[16]. In this paper we present detailed calculations of the electron elastic scattering in an intense magnetic field. The transition rates studied depend on the polarization of the electron spin third component with respect to the direction of the magnetic field. We also consider the situation when the spin points in the direction of motion.

2. MAGNETIC ELASTIC DISPERSION

When the electron collides with an intense magnetic field some of the processes mentioned in the introduction may occur. One of them could be that the electron is dispersed elastically by the magnetic field. The radiation processes of the electron in the magnetic field may be more important than the elastic ones. Nevertheless, the magnetic elastic scattering of electrons is a good example in which we can develop calculation skill utilizing Dirac's exact wave functions according to the Furry's framework[17]. The electron can be scattered elastically one, two, . . . n-times by the magnetic field, see Fig. 1. The convergence of the transitions rates for those processes may be con-

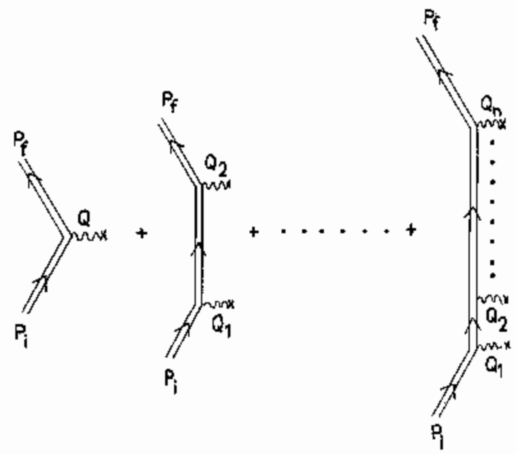

Figure 1

Feynman's diagrams for the elastic scattering of electrons in an external field. The electron is dispersed one time, two times, . . . n—times elastically in the external field.

sidered as a test for the convergence of transitions rates of the radiative processes of higher order. Nevertheless, we should point out that there are no detailed studies about the elastic dispersion of electrons in a magnetic field[18]

The S-matrix element for the first order in elastic scattering is

$$S_{fi}^{(1)} = (-ie) \int d^4 x\ \Psi_{out}(x)\ A^{ext}(x)\,.\ \Psi_{in}(x) \qquad 2.1$$

where both Ψ_{in} and Ψ_{out} are the complete wave functions obtained by solving the Dirac equation with the external magnetic field $A_\mu^{ext}(x)$. The expression, explicitly, for an homogeneous magnetic field along the z–axis is the following

$$A^i = -(B/2)\ \epsilon^{ik}\ x^k\ ,\ \epsilon^{12} = -\epsilon^{21} = 1 \qquad 2.2$$

$$\epsilon^{11} = \epsilon^{22} = 0$$

The incoming and outgoing electron wave functions are

$$\Psi_{in}(x) = (\gamma / L\pi)^{1/2} \, e^{-iE_i t} \, e^{ip_{3i} z} \, X(\rho)\, u^{(r)}(k), \quad 2.3$$

$$\overline{\Psi}_{out}(x) = (\gamma / L\pi)^{1/2} e^{iE_f t} \, e^{-ip_{3f} z} \, . \, \bar{u}^{(s)}(k)\, \overline{X}(\rho),$$

Where $\chi(\rho)$ is a matrix containing the radial part in the x-y plane, plus angular terms of the wave function; it is defined by

$$X(\rho) = \begin{pmatrix} I_{n-1,s}(\rho) e^{i(\ell-1)\phi} & 0 & 0 & 0 \\ 0 & iI_{n,s}(\rho)\, . \, e^{i\ell\phi} & 0 & 0 \\ 0 & 0 & I_{n-1,s}(\rho)\, e^{i(\ell-1)\phi} & 0 \\ 0 & 0 & 0 & iI_{n,s}(\rho)\, e^{i\ell\phi} \end{pmatrix} \quad 2.4$$

where $\rho = \gamma \, . \, r^2$ and $\gamma = eB/2$

In this expression the functions $I_{n,s}(\rho)$ and $I_{n-1,s}(\rho)$ are the generalized Laguerre functions defined as follows.

$$I_{n,s}(\rho) = (s!/n!)\, . \, e^{-\rho/2} . \, \rho^{(n-s)/2} \, L_s^{\,n-s}(\rho) \quad 2.5$$

The spinors $u^{(s)}(k)$ are of two kinds: when the third component of spin the electron points along or opposite to the magnetic field and when the spin orientation points along or opposite to the motion of the electron.

We do not pretend to calculate the sum of the diagrams of the Fig. 1 in order to prove the convergence of the series, but instead we would like to be concerned only with the first term of the serie, which corresponds to the process when the electron is scattered just one time. On the other hand, it will allow us to develop the calculation techniques which are of relevance for the radiative processes in the Furry's framework[19]

A more detailed discussion and the properties of these spinors are presented in the appendix.

We now substitute the above equations in Eq. (2.1) and, through a straightforward calculation, we obtain the following expression for the matrix element

$$S_{fi} = (-ie)(1/L\gamma^{1/2})(B/2)\, |2\pi \, . \, \delta\,(E_i - E_f)| \, [2\pi\, \delta\,(P_{3i} - P_{3f})]\sqrt{n}\; \bar{u}^{(z)}(k')\,\gamma^1 \, . \, u^{(s)}(k) \quad 2.6$$

where

$\Gamma_1 = (1/2)(\gamma^1 + i\,\gamma^2)$ and $\Gamma_2 = (1/2)\, . \, (\gamma^1 - i\,\gamma^2)$

The transition probability for the first-order elastic scattering process, $|S_{fi}|^2$ is thus given by

$$|S_{fi}|^2 = (e^2 / L^2\ \gamma)(B/2)^2\ 2\pi\ T\ \delta\,(E_i - E_f)\, . \, 2\pi\, . \, L\, . \, \delta\,(P_{3i} - P_{3f})\, . \, n\ \bar{u}^{(r)}(k')\,\gamma^1\, u^{(s)}(k)\ \bar{u}^{(s)}(k)\,\gamma^1 \, . \, u^{(r)}(k') \quad 2.7$$

The two delta functions contained in $|S_{fi}|^2$ correspond to the conservation of energy and to the third component of momentum.

3. ELASTIC SCATTERING WITH POLARIZATION PARALEL TO THE MAGNETIC FIELD

Eq. (2.8) is so general, that for the case we are concerned with we just have to substitute the corresponding spinors in order to analyze the cross section derived from it. Thus, taking into account the orientation along the magnetic field of the incoming and outgoing third component of the electron spin, the differential cross sections are going to be given in terms of $|S_{fi}|^2$ and the spin orientation by the expressions

$$d\sigma^{up,up} = (m^2/8\ \pi^2)(B/B_{cr})(P_i^2 / |\vec{P}_i|^2)\ n\ |1 + (E_i + m)^2 / (P_i^2\ \cos\Phi + 2m(E_i + m) + P_{i\,3}^{\,2})| \quad 3.1$$

where $B_{cr} = m^2\ c^3 / \hbar e = 4.414.10^{13}$ Gauss is the critical field. Moreover we have used the following kinematical relations under a cylindrical symmetry of the magnetic field and just considering that the collision takes place at the origen of coordinates

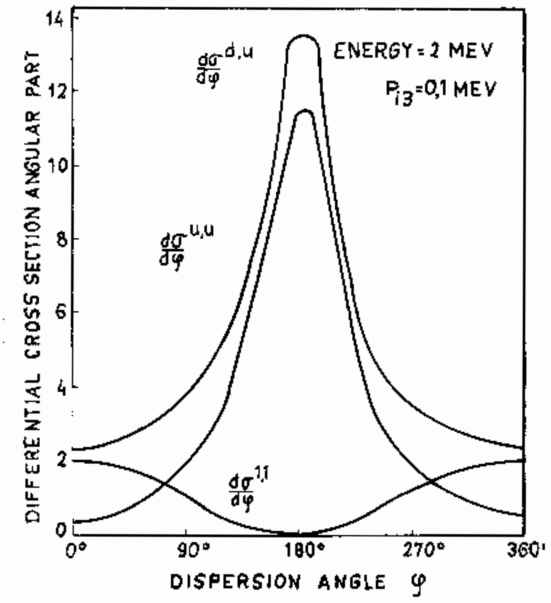

Figure 2

The differential cross section angular part of an electron dispersed elastically by an external field. The energy of the electron is 2 Mev. with a small momentum third component of. 1 Mev.

$$d\sigma^{\,down,up} = (m^2 / 8\pi^2)(B/B_{cr})(P_{i3}{}^2 / |\vec{p}_i|^2)\, n \,|-1 + (E_i + m)^2 / (P_i{}^2 \cos\Phi + 2m(E_i + m) + P_{i3}{}^2)| \qquad 3.2$$

where $B_{cr} = m^2 c^3 / \hbar e = 4.414.10^{13}$. Gauss is the critical field. Moreover we have used the following kinematical relations under a cylindrical symmetry of the magnetic field and just considering that the collision takes place at the origen of coordinates

$$E_i{}^2 = E_f{}^2 = P_{i1}^2 + P_{i2}^2 + P_{i3}^2 + m^2 = m^2 + P_i{}^2 + P_{i3}{}^2 = m^2 + P_{i3}^2 + 4\gamma n \qquad 3.3$$

$$P_{i1} = P_i \cos\Psi \quad P_{f1} = P_f \cos\alpha$$

$$P_{i2} = P_i \sin\Psi \quad P_{f2} = P_f \sin\alpha$$

and setting $\rho = \Psi - \alpha$ as the dispersion angle of the electron. It can easily be proved that Eq. 3.1 and Eq. 3.2 also fullfill the relations

$$d\sigma^{\,up,\,up} = d\sigma^{\,down,\,down}$$
$$d\sigma^{\,up,\,down} = d\sigma^{\,down,\,up} \qquad 3.4$$

Now, for the case when we are not interested in the dependency of the cross section on the spin orientation, we must sum over the incident spin polarization and average over the outgoing spin orientation of the electron, thus obtaining.

$$(1/2)\Big(\sum_{\pm S_f \,\pm S_i} |S_{fi}|^2\Big) = (e^2 / L^2\gamma)\,(B/B_{cr})^2\; 2\pi\; T\,\delta\,(E_i - E_f)\;(2\pi L)\;\delta\,(P_{i3} - P_{f3})\,(n/2)\sum_{\pm S_{fi}\,\pm S_i} \bar{u}^{(r)}(k')\,\gamma^1\, u^{(s)}(k)\,\bar{u}^{(s)}(k)\;\gamma^1\; u^{(r)}(k') \qquad 3.5$$

By constructing the differential cross section as Bjorken and Drell indicate *, we get

$$\omega = d\sigma/d\phi = 1/2 \int_{\pm S_f,\,\pm S_i} \Sigma\; |S_{fi}|^2\, V^2\, E_{i3}\, E_f\, dE_f\, dp_{f3} / |(2\pi)^3\, T\; |\vec{P}_i|\,| = (m^2/8\pi)(B/B_{cr})\,P_i{}^2 / |\vec{P}_i|\, n\,(1 + \cos\phi) \qquad 3.6$$

We should remember that all the above equations are valid only under the assumption that the quantum number S and S' be equal: S = S'.

4. MAGNETIC ELASTIC SCATTERING WITH LONGITUDINAL POLARIZATION

In describing the scattering of electrons in a magnetic field we may consider that the spin of the incident electron points in the direction of motion so that we may expect to get additional

* The notation and metric of Bjorken and Drell[14] are used throughout these calculations. We employ a system of units in which $\hbar = c = 1$.

N.B.: For simplicity in the calculations we do not construct a uniform beam by taking an unweighted average over the guiding center of the incident electron (see (1): Langer SH).

information about the elastic scattering. There are some differences with respect to the case of the preceding section. Specially the bounded wave function has different spinors which caracterize the two possibilities of orientation of the spin: along the motion and opposite to it.

Also, from now on, we can study, with the help of Eq. 2.8 the transition probabilities, according to the spin polarizations of the incoming and outgoing electron. The kinematic relations for those processes is given by Eq. 3.3.

By constructing the differential cross section when the incoming and outgoing spin of the electron points in the direction of motion, we get

$$d\sigma^{1,1}/d\phi = \int L\ |S_{fi}|^2\ V^2\ E_i\ E_f\ dE_f\ dp_{f3} / [V^2\ T(2\pi)^3\ |\vec{P}_i|] = (e^2 B^2 / 16\pi\gamma)((p_i)^2 / [\vec{P}_i])(n)\ (1 + \cos\phi) \qquad 4.1$$

If the outgoing electron leaves the magnetic field with a spin orientation against the motion, the cross section is

$$d\sigma^{1,2}/d\phi = 0 \qquad 4.2$$

It can easily be proved that the cross section, for the case when the electron enters into the magnetic field with spin orientation against the motion and leaves it with spin oriented along the motion, is also zero:

$$d\sigma^{2,1}/d\phi = 0 \qquad 4.3$$

and the cross section also satisfies the relation

$$d\sigma^{1,1}/d\phi = d\sigma^{2,2}/d\phi = (m^2/8\pi)(B/B_{cr})\ P_i^2 / |\vec{P}_i|\ n(1 + \cos\phi) \qquad 4.4$$

An evident conclusion, from the above equations, is that, if we consider the polarization of the electron along the motion in the first order of the elastic scattering process, there is not a spin-flip. We should remark also that, from the preceding paragraph, the question arises of how can an electron be scattered when its motion is along the magnetic field. We do not pretend to answer it, but we leave it as an open problem to be analized some other time. For the case in which we sum over the incident spin polarization and average over the outgoing spin orientation of the electron, we get

$$d\sigma/d\phi = (m^2/32\pi)\ (B/B_{cr})\ (P_i^2 / |\vec{P}_i|)\ n\ (1 + \cos\phi) \qquad 4.5$$

It is remarkable that the cross sections for the first order elastic scattering are different from zero and that tells us that, although they may be small compared with the radiation processes for the same energy of electron, they exist and are finite.

It is one fourth of the cross section when we take into account the spin orientation either along the motion or against it.

5. CONCLUSIONS

Furthermore, the total quantum number must be different from zero; if it is zero, then $P_i = P_f = 0$ and we do not have scattering at all. If the electron has a small third component of the momentum, there is a probability for the ocurrence of spin flip in the case when the spin points along the magnetic field. If the spin is along the motion this phenomen does not occur.

By comparing both cases, when the electron spin is along the magnetic field and the spin is along the motion, and we do not care about the spin polarization, the cross section for the first case is four times greater than the second one. Both cross sections have the same shape but the second is just smaller by a factor of four.

Due to the fact that maximum of the cross section appears when the dispersion angle is zero, the magnetic field behaves as a medium of different refraction index, so that a ray of electrons is refracted by the magnetic field and the ray emerges somewhere but making the same angle with respect to the incident one. This is self-evident because of the cylindrical symmetry of this problem and only if we consider the motion in the plane x-y.

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18. Idem.

19. Idem.

APPENDIX: WAVE FUNCTIONS AND SPINOR ALGEBRA

a) Third component spin polarization along the magnetic field. In order to get a better insight on the solutions of Dirac equation [13]

$$(i\not\partial - eA - \chi)\,\Psi = 0$$

with the vector potential of the form $A_i = -\frac{B}{2}\,\epsilon_{ik}\,x_k$, we consider the wave function for the particle according to the spin orientation along the magnetic field in the following form:

Spin up: it means that the third component is oriented along the magnetic field

$$\Psi^{(1)}(\vec{r},t) = \sqrt{(\gamma/L\pi)}\; e^{-iEt}\, e^{ip_3 z}\,\chi(\rho)\,u^{(1)}(\vec{p}) \qquad \text{A.1}$$

Spin down: it means against the magnetic field:

$$\Psi^{(2)}(\vec{r},t) = \sqrt{(\gamma/L\pi)}\; e^{-iEt}\, e^{ip_3 z}\,\chi(\rho)\,u^{(2)}(\vec{p}) \qquad \text{A.2}$$

where $\chi(\rho)$ is defined in Eq. (2.4), $u^{(1)}(p)$ and $u^{(2)}(p)$ are the bound-state spinors defined as follows

$$u^{(1)}(\vec{p}) = 1/\sqrt{2E(E+m)}\begin{pmatrix} E+m \\ 0 \\ K_3 \\ \sqrt{4\gamma n}\end{pmatrix}$$

A.3

$$u^{(2)}(\vec{p}) = 1/\sqrt{2E(E+m)}\begin{pmatrix} 0 \\ E+m \\ \sqrt{4\gamma n} \\ -K_3\end{pmatrix}$$

The indixes appearing in the matrix $\chi(\rho)$; n, ℓ, are the total quantum numbers, respectively azimutally and S is related with them through $S = n - \ell$. L is the periodical length and finally

$$\gamma = eB/2$$

The solutions with negative energy corresponding to the antiparticle are

$$\Psi^{(3)}(\vec{r},t) = C\,\overline{\Psi}^{T(1)}$$

$$\Psi^{(4)}(\vec{r},t) = C\,\Psi^{T(2)} \qquad \text{A.4}$$

with $C = i\gamma^2 . \gamma^0$. This allows us to obtain the antiparticle spinors

$$V^{(1)}(p) = 1/\sqrt{2E(E+m)}\begin{pmatrix} \sqrt{4\gamma n} \\ -K_3 \\ 0 \\ E+m\end{pmatrix}$$

spin down, A.5

$$V^{(2)}(p) = 1/\sqrt{2E(E+m)}\begin{pmatrix} K_3 \\ \sqrt{4\gamma n} \\ E+m \\ 0\end{pmatrix}$$

spin up. A.5

For calculations with the above wave functions it is usefull to keep a list of relations between the spinors. They satisfy the following ortogonality relation

$$\bar{u}^{(r)}(p)\,u^{(s)}(p) = (m/E)\,(\delta_{rs})$$

A.6

$$\bar{V}^{(r)}(p)\,V^{(s)}(p) = -(m/E)\,(\delta_{rs})$$

The free spinors can be obtained from the above ones just by letting $B \rightarrow 0$. The ortogonality is then

$$\bar{u}^{(r)}(0)\ u^{(s)}(0) = \delta_{rs}$$

$$\bar{V}^{(r)}(0)\ V^{(s)}(0) = -\delta_{rs} \qquad \text{A.7}$$

which is in accordance with the plane Q.E.D. (Bjorken and Drell)[14]. By relating the free spinors and the bound–state spinors we have

$$u^{(r)}(p) = [\, m + \gamma^0 E - \gamma^3 P_3 - \gamma^1 (\sqrt{4\gamma n})\,] / \sqrt{2E(E+m)}\,.\,(u^{(r)}(0)) \qquad \text{A.8}$$

$$v^{(r)}(p) = [m - \gamma^0 E + \gamma^3 P_3 + \gamma^1 (\sqrt{4\gamma n})] / \sqrt{2E(E+m)}\,.\,(V^{(r)}(0))$$

We can also construct proyection operators as follows

$$\Lambda_+(p) = \bar{u}^{(1)}(p)\,.\,u^{(1)}(p) + \bar{u}^{(2)}(p)\,.\,u^{(2)}(p) = (1/2E)(m + \gamma^0 E - \gamma^3 P_3 - \gamma' \sqrt{4\gamma n})$$

$$\text{A.9}$$

$$\Lambda_-(p) = \bar{V}^{(1)}(p)\,.\,V^{(1)}(p) + \bar{V}^{(2)}(p)\,.\,V^{(2)}(p) = (-1/2E)(m - \gamma^0 E + \gamma^3 P_3 + \gamma^1 \sqrt{4\gamma n})$$

If we apply $\Lambda_\pm(p)$ to the solution of the Dirac equation one obtains only solutions with positive frecuency (particle) or only solutions with negative frecuency (antiparticle); it means

$$\Lambda_+(p)\, u^{(r)}(p) = (m/E)\, u^{(r)}(p)$$

$$\Lambda_-(p)\, V^{(r)}(p) = (m/E)\, V^{(r)}(p) \qquad \text{A.10}$$

These proyection operators satisfy the following relations

$$\Lambda_\pm(p)\,\Lambda_\pm(p) = 0$$

$$(\Lambda_+(p))^2 = (m/E)\,\Lambda_+(p)$$

$$\Lambda_+(p) + \Lambda_-(p) = m/E \qquad \text{A.11}$$

$$(\Lambda_-(p))^2 = -(m/E)\,\Lambda_-(p)$$

$$\sum_{S=1}^{2} (u_\alpha^{(s)}(p)\, \bar{u}_\beta^{(s)}(p) - V_\alpha^{(s)}(p)\, \bar{V}_\beta^{(s)}(p)) = (m/E)\,(1)_{\alpha\beta}$$

From the above we notice that the solutions for the particle, i.e., for the antiparticle, alone, do not form a complete system, this because the operators $\Lambda_\pm(p)$ are not equal to the unity matrix and they have rather the properties of proyection operators.

Some others useful rules for these operators are the following:

$$\gamma_0 \Lambda_\pm^+(p)\,.\,\gamma_0 = \Lambda_\pm(p)$$

$$\Lambda_+(p)\,\Lambda_\pm^+(p) = \pm(E/m)\,\gamma^0\,\Lambda_\pm^+(p)$$

$$\Lambda_\pm(p)\,\gamma^0\,\Lambda_\pm^+(p) = \gamma^0\,\Lambda_\pm^+(p) \qquad \text{A.12}$$

$$u^{+(s)}(p)\,u^{(r)}(p) = (E/m)\,\delta_{sr}$$

$$V^{+(s)}(p)\,V^{(r)}(p) = -(E/m)\,\delta_{sr}$$

b) Spin polarization along the motion.

We consider the spinors of the electron along the motion, which are: Spin orientation along the direction of motion

$$u^{(1)}(k) = 1/(2\sqrt{E\,p}) \begin{pmatrix} \sqrt{(E+m)(p+k_3)} \\ \sqrt{(E+m)(p-k_3)} \\ \sqrt{(E-m)(p+k_3)} \\ \sqrt{(E-m)(p-k_3)} \end{pmatrix} \qquad \text{A.13a}$$

Spin orientation against the direction of motion,

$$u^{(2)}(k) = 1/(2\sqrt{Ep}) \begin{pmatrix} \sqrt{(E+m)(p-k_3)} \\ \sqrt{(E+m)(p+k_3)} \\ \sqrt{(E-m)(p-k_3)} \\ \sqrt{(E-m)(p+k_3)} \end{pmatrix} \qquad \text{A.13b}$$

The spinors of the antiparticle follow form (A.4), and we get

$$V^{(1)}(k) = 1/(2\sqrt{E\,p}) \begin{pmatrix} \sqrt{(E-m)(p+k_3)} \\ \sqrt{(E-m)(p-k_3)} \\ \sqrt{(E+m)(p+k_3)} \\ \sqrt{(E+m)(p-k_3)} \end{pmatrix}$$

A.13c

$$V^{(2)}(k) = 1/(2\sqrt{E\,p}) \begin{pmatrix} \sqrt{(E-m)(p-k_3)} \\ -\sqrt{(E-m)(p+k_3)} \\ -\sqrt{(E+m)(p-k_3)} \\ \sqrt{(E+m)(p+k_3)} \end{pmatrix}$$

A.13d

The satisfy the following ortogonality relation

$$\bar{u}^{(r)}(k)\,u^{(s)}(k) = (k_0 / k)\,\delta_{rs}$$

A.14

$$\bar{V}^{(r)}(k)\,V^{(s)}(k) = -(k_0 / k)\,\delta_{rs}$$

The proyection operators constructed with these operators are the same as the expressed in Eq. A.9. The relations derived for these spinors satisfy also Eq. A.10, Eq. A.11 and Eq. A. 12.