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MOTION IN MELVIN METRIC

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RESUMEN

Se presenta un análisis cualitativo y numérico que refleja cómo el fotón y la partícula de masa con carga y sin ella se mueven bajo la influencia pura de la métrica cilíndrico-magnética de Melvin.

ABSTRACT

A qualitative and a numerical analysis is presented showing how the photon and charged and uncharged particle with mass move under the influence of Melvin's pure cylindrical magnetic metric.

1. INTRODUCTION

The magnetic fields play an important and decisive role in the possibility of observation of the radiation of some massive stellar objects like the neutrons stars, pulsars and black holes. Since the discovery of the pulsar stars there has been a rapid growth in astrophysics in reference to the effects of superstrong magnetic fields ($B=10^{12}$ Gauss) on the radiative quantum electrodynamics processes. Among these effects the most significant are those involving the emission of X-ray and γ-radiation, especially synchrotron emission. Those processes may be the cause of the hard radiation observed in great variety of collapsing stellar objects.

The purpose of this paper is not to calculate some radiative processes in superstrong magnetic fields (this will be reported elsewhere), but instead to provide a better insight of the behavior of classical motion of charged and uncharged particles under the influence of extremely strong magnetic fields, such as those existing in pulsar stars (or neutron stars). This will in principle allow us to better understand the radiative processes occurring under such extreme conditions.

The only analytical model which reproduces the strong influences of the magnetic field lines in a limited region space is the Melvin Universe and it will be used as a first approach to the problem. It is the simplest model of universe including this interaction. The Melvin universe is a formal solution of the Einstein-Maxwell equations and corresponds to a classical description of magnetic field lines parallel to each other and in equilibrium under their mutual gravitational attraction. The paths of photons, charged and uncharged particles in this universe are examined. Section 2 presents a description of the Melvin universe; in section 3 presents the equation of motion for the particles subject to the Lorentz force. In section 4 contains

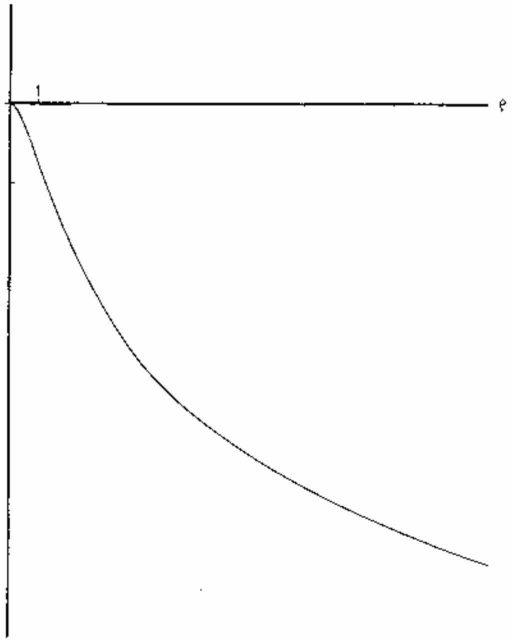

Figure 1

The newtonian potential Ψ associated to the Melvin metric is displayed as a function of ρ.

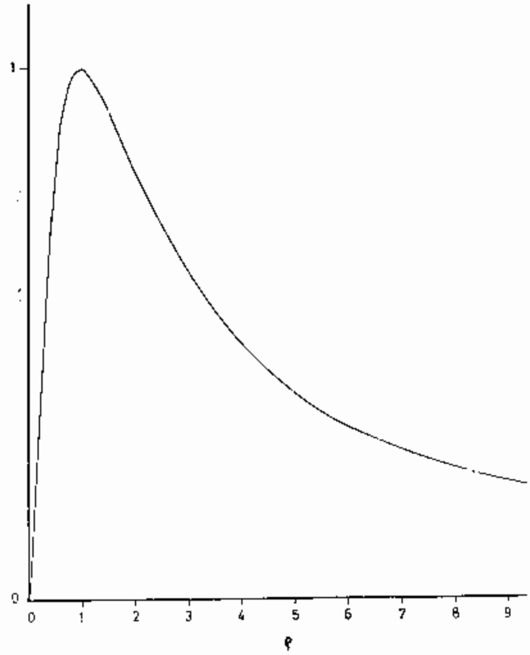

Figure 2

The acceleration of gravity for the Melvin universe as a function of ρ. With ρ = 1 it has a maximum of acceleration.

quantitative analysis of the possible paths of the particles in this universe. In section 5 discusses the numerical results obtained by solving the equations of motions and compare our results with those of the Ernst metric [5,6].

2. THE MAGNETIC UNIVERSE OF MELVIN

This section presents the principal characteristics of the Melvin universe. The geometry of the cylindrical magnetic universe is give by.

$$dS^2 = \Delta^2 \{ -(d\,ct)^2 + d^2\rho + dz^2 \} + \rho^2/\Delta^2 \; d\psi^2 \qquad 2.1$$

where

$$\Delta := 1 + GB_0^2\,\rho^2 \,/\, 4c^4 \qquad 2.2$$

and B_0 is the magnetic field strength along the symmetry and axis, measured in Gauss units, G is the gravitational constant and c is the velocity of light.

The newtonian potential associated to this metric is

$$\Psi = -\ln(1 + GB_0^2\,\rho^2 \,/4c^4) \qquad 2.3$$

In Fig. 1 displays the dependence of Ψ as a function of when $GB_0^2 / 4c^2 = 1$. Notice that the acceleration of gravity has a maximun atρ_0=1. The acceleration of gravity is represented in Fig. 2. The magnetic field for this universe is given by

$$B = B_0/\Delta^2 \qquad 2.4$$

and it is shown in Fig. 3. Its asymptotic behaviour indicates that when $\rho \to \infty$then $B \to 0$, but slowly. This means that the magnetic field exits in every point of the universe. In order to understand better this universe, will be studied the classical behaviour of charged particles. The analysis of the equations of motions can be done three ways: by solving the Lorentz equations, by solving the Euler-Lagrange equations and by solving the Hamilton-Jacobi equations. All of them are equivalent. The use of the Lorentz equations was chosen because of simplicity.

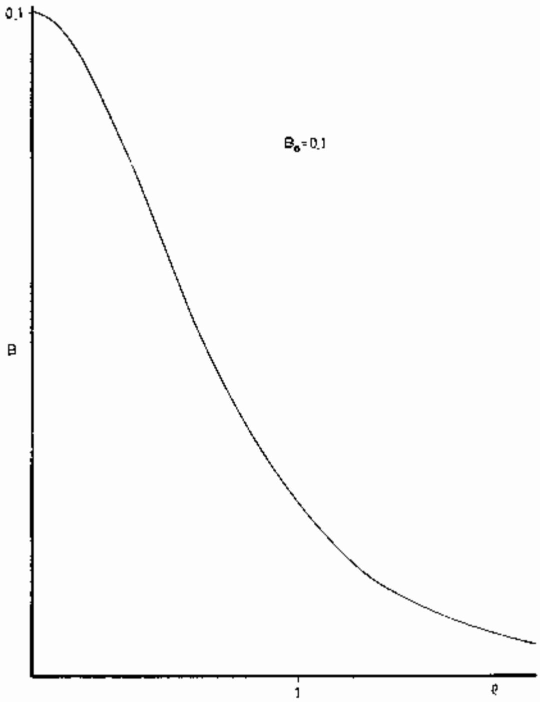

Figure 3

The behaviour of the magnetic field as a function of ρ.

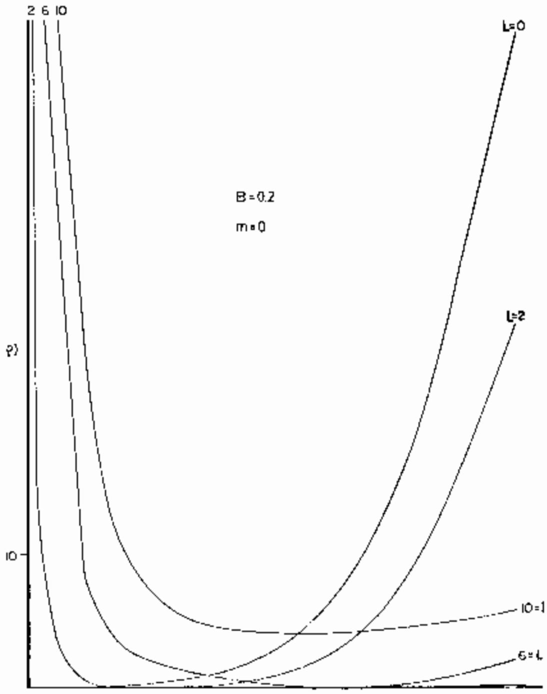

Figure 4

The behaviour of the effective potencial R (ρ) as a function of the variable ρ. This function is displayed for the external field B=0.2 and m=0 and different angular momenta.

3. THE EQUATIONS OF MOTION

The motion of a test charged particle in the Melvin universe can be determined by solving the Lorentz equations of motion. By knowing the corresponding expression for $F\,\mu\nu$ and the Cristoffel symbols for the Melvin metric, the following coupled system of equations is obtained:

$$d^2\tau / d\lambda^2 + (B^2 . \rho / \Delta) . (d\tau / d\lambda) (d\rho / d\lambda) = 0$$

$$d^2 z / d\lambda^2 + (B^2 . \rho / \Delta) . (dz / d\lambda) . (d\rho / d\lambda) = 0 \qquad 3.1$$

$$d^2\rho / d\lambda^2 + (B^2 \rho / 2\Delta) . (d\rho / d\lambda)^2 + (B^2 \rho^3 / 2\Delta^5 - \rho / \Delta^4) . (d\phi / d\lambda)^2 - - (B^2 \rho / 2\Delta) . (dz / d\lambda)^2 + (B^2 . \rho / 2\Delta) . (d\tau / d\lambda)^2 = (eB\rho / m^2 c^2 \Delta^4) . d\phi / d\lambda$$

$$d^2\phi / d\lambda^2 + 2 (1/\rho - B^2 \rho / 2\Delta) . (d\phi / d\lambda) (d\rho / d\lambda) = -(e / mc^2) . (B/\rho)(d\rho / d\lambda)$$

where λ is an affine parameter. These equations are consistent with the classical equations of motion of a charged particle in a constant uniform magnetic field wich points along the z-axis. This can be easily be proved by putting $B^2 \rightarrow 0$ if the affine parameter is the time τ . It implies that the magnetic field is weak in this universe.

The first integrals of the above system of equations can be found analytically and they are:

$$d\tau / d\lambda = E / mc . \Delta^2$$

$$dz / d\lambda = p_z / m . \Delta^2$$

$$d\phi / d\lambda = [L/m - (eB/mc^2) . (\rho^2 / 2\Delta)] \Delta^2 / \rho^2 \qquad 3.2$$

$$d\rho / d\lambda = 1 / \Delta^2 [E / m^2 c^2 - [\Delta^2 c^2 + p_z^2 / m^2 + (\Delta^4 / \rho^2) . (L/m - (eB / mc^2) . (\rho^2 / 2\Delta)]]^{1/2}$$

where L is the angular momentum of the particle.

The integration of the last two equations give elliptic functions which provide little insight. In order to more easily analyze the results, they are instead solved numerically.

4. THE QUALITATIVE ANALYSIS OF THE PATHS

Some results can be easily obtained from a qualitative analysis "a la Poincare". First, define an effective potential as

$$V(\rho) = m^2 . \Delta^2 + p_Z^2 + (\Delta^4 / \rho^2) . (L - (eB / 2 \Delta) . \rho^2)^2 \quad 4.1$$

and the function

$$R(\rho) = [E^2 - V(\rho)]^{1/2} \quad 4.2$$

Next, introduce a new function

$$U^2 := E^2 - p_Z^2 \quad 4.3$$

that to be called the transversal energy. For the sake of simplicity the following cases are considered:

1) The motion of uncharged particles with rest mass equal to zero (the photon).

The corresponding equation of motion for the photon is

$$d\phi / d\rho = L . \Delta^4 / (\rho^2 \sqrt{E^2 / c^2 - \Delta^4 L^2 / \rho^2 - p_Z^2}) \quad 4.4$$

This equation admits six other posibilities, they are:

i) If B=0, then the motion of the photon is along a straight line. This corresponds to the motion of a photon in a Minkowski space. This motion is invariant under a rotation of x-y axis.

ii) If L =0, then $d\phi / d\rho = 0$. This implies also a motion along a straight line.

iii) If $d\phi / d\rho = 0$, then we can get the critical points of the path of the photon. By ii) one of the solution is that L=0, it means that there are not critical points and there is straight line motion.

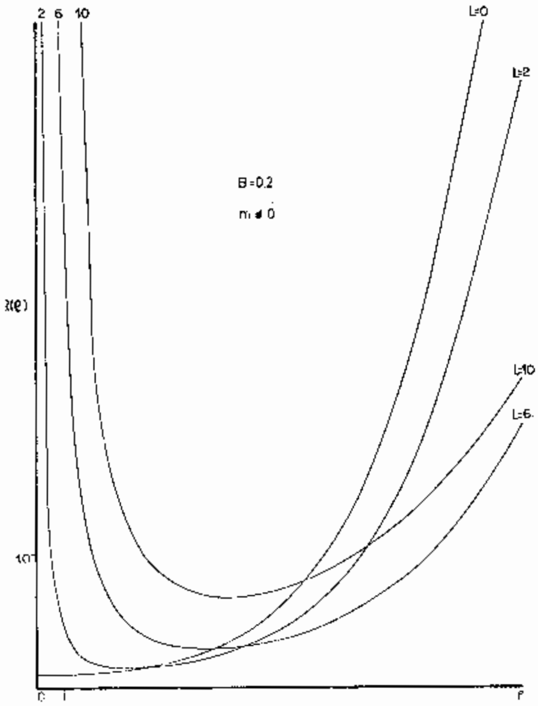

Figure 5

The effective potential R (ρ) as a function of ρ with m $\neq$ 0. For this calculation m=1 and e=1.

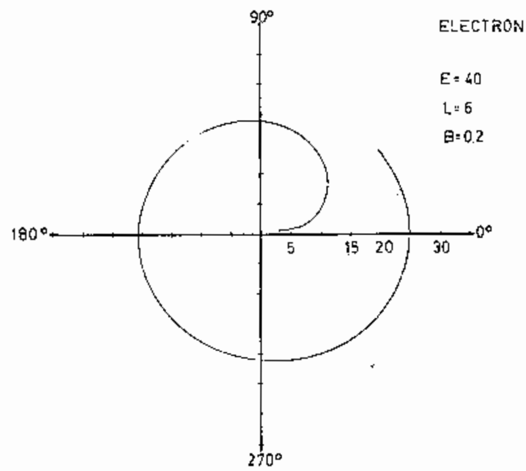

Figure 6

The path of an electron under the influence of the pure magnetic metric of Melvin. The energy is 40 units, the angular momentum is 6 and the external field strenght is 0.2. We used G=c=1.

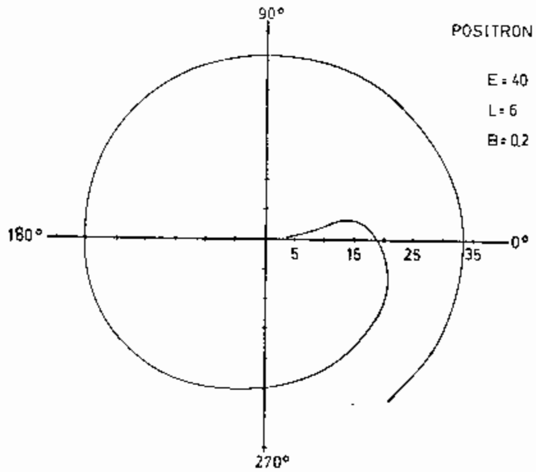

Figure 7

The path of a positive charged particle (positron) moving in the universe of Melvin.

Also the solution $\Delta^4 = 0$ means that there are not critical points either.

iv) a) A helical motion results from setting $\rho =$ = constant. The radius of the orbit can be calculated by setting $R(\rho) = 0$.

b) A circular motion exists if $p_Z = 0$.

v) If $U^2 = 0$, then equation (4.4) is reduced to

$$d\phi / d\rho = L \Delta^4 / \sqrt{-\Delta^4 L^2 \rho^2} \qquad 4.5$$

This is pure imaginary for every $\rho \neq 0$. It implies no motion parallel to the z-axis for $L \neq 0$. If $\rho = 0$ them it satisfies point a) from above and it implies that L must be equal to zero and it admits motion on the z-axis.

vi) The numerical calculation were made by taking the adiabatic approximation of equation (4.4). The result is shown in Fig. 8.

2) The motion of uncharged particles

The equation of motion for uncharged particles is

$$d\phi / d\rho = L \Delta^4 / (\rho^2 \,.\, \sqrt{E^2/c^2 - [m^2 c^2 \,.\, \Delta^2 + p_Z^2 + (\Delta^2/\rho^2) \,.\, L^2]}) \qquad 4.6$$

This equations admits the following possibilities, they are:

i) If B=0, then the motion is along a straight line. It means the motion of a neutral particle in Minkowski space-time.

ii) If L=0, then the motion is also a straight line.

iii) The critical points can be calculated by letting $d\phi / d\rho = 0$

The solutions are two: one for L=0 and the other one is for $B^2 \rho^2 = -4$. Both solutions implied that there are not critical points.

iv) a) The helical motion exists if ρ =constant. By letting $R(\rho) = 0$, we get the radius of the orbit.

b) The circular motion exists if $p_Z = 0$

v) For the motion parallel to the z-axis we put $E^2/c^2 - p_Z^2 - m^2 c^2 = 0$. Then the expression $d\phi / d\rho$ is pure imaginary. It implies no parallel motion to the z-axis. Because of the condition imposed in point a) from above if L=0 there is motion on the z-axis.

vi) A meridian motion exists if $E^2 / c^2 - p_Z^2 - m^2 \,.\, \Delta^2 \,.\, c^2 = 0$. This implies that $\rho_0 = 0$. It permits

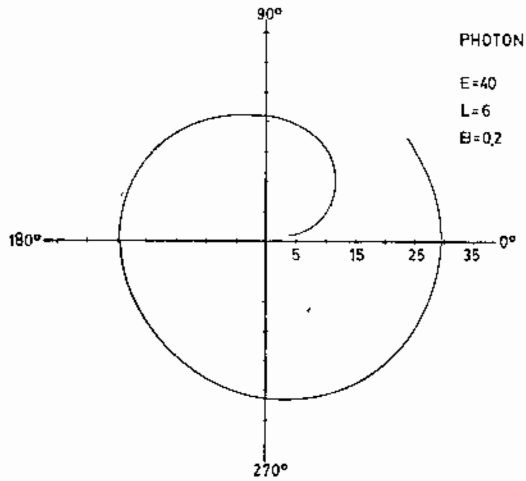

Figure 8

The photon is also curved by Melvin's universe.

just motion on the z-axis for uncharged particles with $m \neq 0$ and $L = 0$.

3) The motion for charged particles

The corresponding equation for the motion of charged particles is

$$d\,\phi / d\,\rho = Q(\rho)\,.\,\Delta^4 / (\rho^2\,.\,\sqrt{E^2/c^2 - [m^2\,c^2\,\Delta^2 + p_z^2 + (\Delta^4/\rho^2)\,.\,Q^2(\rho)\,]}\,) \qquad 4.7$$

where

$$Q(\rho) = L \pm (eB/mc^2)\,.\,\rho^2/2\Delta \qquad 4.8$$

This equation admits the following possibilities:

i) If $B=0$ the motion is a straight line. This can easily be proved numerically and it corresponds to the motion of charged particles in Minkowski space–time under the influence of an extended magnetic field given by (2.4).

ii) If $L=0$, then the motion is not longer a straight line.
This follows from the numerical calculation.

iii) The critical points can be calculated by letting $d\,\phi / d\,\lambda = 0$.
This gives the following condition that must be fullfilled

$$\rho = \sqrt{-2\,c^2\;L\,(LB^2\,c^2/_2 \pm eB)} \qquad 4.9$$

iv) a) The helical motion exists if ρ = constant. The radius of the orbit is obtained by setting $R(\rho) = 0$. It means that

$$E^2/c^2 - [\,m^2\,.\,\Delta^2\,c^2 + p_z^2 + (\Delta^4/\rho^2)\,.\,Q^2(\rho)\,] = 0 \qquad 4.10$$

b) The circular motion parameters are determined just by putting $p_z = 0$ in the equation from above.

v) The motion parallel to the z-axis impose the condition that $E^2 - [\,m^2\,.\,\Delta^2\,c^2 + p_z^2\,] = 0$. It implies that $d\,\phi/d\,\rho$ is pure imaginary for every $\rho \neq 0$. If $\rho = 0$ then it satisfies point

a) from above and it follows then that if $L=0$ there is motion on the z-axis only.

5. CONCLUSION AND REMARKS

As was pointed out in the introduction, it should be valuable to know about the motion of charged and uncharged particles under the influence of superstrong magnetic fields. As is known from resent results from astrophysics, the pulsars or the neutron stars have magnetic field of the order $B=0.1\,B_{cr}$, where B_{cr} is the critic magnetic field of Schwinger ($B_{cr} = m^2\,c^3/e\hbar = 4.414.10^{13}$ Gauss). This makes it appear that charged particles in the accreation column of those stars are subject to huge magnetic fields. Near the surface of the star the influence of the gravitational field is also the extremely large. Not considered in this analysis is the presence of the mass of the star which also influences the motion of the particles. However in a region near the surface of the stars the conditions for the existence of such magnetic field may also be fullfilled. Nevertheless, a study including gravitational effects has been done by Dadhich et al. (3) For a numerical analysis suppose that $p_z = 0$. Fig. 4 and fig. 5 show the effective potential for different angular momentum. Fig. 6 and fig. 7 analyze the paths of the charged particles for the same angular momentum and energy. By comparision with the calculation of Dadhich et al (3) and Esteban (7) it is noted that the Melvin metric, i.e, the influence of the magnetic field alone is not a quantity to be neglected, although it is smaller compared with the influence of the mass on the path of the particle. As an additional remark, if the field strenght is less than $B < 0.05.B_{cr}$, it makes no difference to use the Melvin or the Minkowsky metric. This may have further consequences for the calculation of radiative processes under the influence of strong magnetic fields.

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